Pergamon

J. Appl. Maths Mechs, Vol. 62, No. 3, pp. 473–475, 1998

0021–8928/98/$—see front matter

PII: S0021–8928(98)00059-8

ON MOTION OF A SYMMETRIC GYROSTAT IN A NEWTONIAN FORCE FIELD†

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(*Received* 21 *March* 1996)

A spherical cavity in a sphere-shaped gyrostat contains a spherical rotor, which is rotating at a constant angular velocity relative to the outer sphere. The centres of the outer sphere, the cavity and the rotor coincide. Attached to the outer sphere are identical point masses, placed at the vertices of an octahedron. A study is presented of the influence of the rotation of the rotor on the existence and stability of steady motions of the gyrostat about its mass centre in the Newtonian field of a fixed attracting centre. Interest centres on motions in which the radius vector of the gyrostat centre and the gyrostatic moment are collinear. It is shown that the existence of a gyrostatic moment may essentially modify the stability properties of the steady motions discovered.

1. POTENTIAL ENERGY. EQUATIONS OF MOTION. FIRST INTEGRALS

Let $Ox_1x_2x_3$ be a system of coordinates rigidly attached to the gyrostat in such a way that the vertices of the octahedron at which the masses m are concentrated have the following coordinates [1]: $A_1 = (1, 0, 0)a$, $A_2 = (-A, 0, 0)a$, $A_3 = (0, 1, 0)a$, $A_4 = (0, -1, 0)a$, $A_5\ (0, 0, 1)a$, $A_6 = (0, 0, -1)a$.

We shall assume that the moment of inertia of the sphere is I_s, and the gyrostatic moment vector has the form $\mathbf{K} = (k_1, k_2, k_3)$. Then, if the body is moving in the Newtonian field of a fixed attracting centre N of mass M, the potential energy is

$$U = -fmM\sum\frac{1}{|\overrightarrow{NA_i}|} = -fmM\sum[R^2 + 2Ra(\gamma, \mathbf{e}_i) + a^2]^{-1/2} \tag{1.1}$$

$$R = |\overrightarrow{NO}|, \quad \gamma = (\gamma_1, \gamma_2, \gamma_3) = \frac{1}{R}\overrightarrow{NO}, \quad \mathbf{e}_i = (e_{1i}, e_{2i}, e_{3i}) = \frac{1}{a}\overrightarrow{OA_i}$$

The equations of motion may be written as

$$\frac{d(I\omega + \mathbf{K})}{dt} = (I\omega + \mathbf{K})\times\omega + \gamma\times\mathbf{U}_\gamma, \quad \frac{d\gamma}{dt} = \gamma\times\omega; \quad I = I_s + I_p; \quad \mathbf{U}_\gamma = \frac{\partial U}{\partial\gamma} \tag{1.2}$$

where I_p is the central moment of inertia of the system of point masses. If the gyrostatic moment vector $\mathbf{K}$ is constant, then, since the inertia tensor I is spherical, the equations of motion may be written as

$$\frac{dI\omega}{dt} = (I\omega + \mathbf{K})\times\omega + \gamma\times\mathbf{U}_\gamma, \quad \frac{d\gamma}{dt} = \gamma\times\omega$$

Equations (1.2) admit of three first integrals: the energy integral $Z_0 = (I\omega, \omega)/2 + U(\gamma) = h$ the projection of the kinetic angular momentum on the γ axis $Z_1 = (I\omega + \mathbf{K}, \gamma) = P_\psi$, and a geometric integral $Z_2 = \gamma^2 = 1$. One first integral is missing for the equations of motion to be integrable.

2. STEADY MOTIONS

We consider the set of steady motions of the gyrostat. It is defined as the set of all critical points of the energy integral at a common level of the integrals Z_1 and Z_2. Let

$$W_{\lambda,\mu} = Z_0 - \lambda(Z_1 - P_\psi) + \frac{1}{2}\mu(\gamma^2 - 1)$$

where λ and μ are Lagrange multipliers.

†*Prikl. Mat. Mekh.* Vol. 62, No. 3, pp. 512–514, 1998.

Then the set of steady motions is found from the system of equations

$$\frac{\partial W_{\lambda,\mu}}{\partial \omega} = I\omega - \lambda I\gamma = 0,\ \frac{\partial W_{\lambda,\mu}}{\partial \gamma} = -\lambda(I\omega + \mathbf{K}) + \mathbf{U}_\gamma + \mu\gamma = 0 \tag{2.1}$$
$$\frac{\partial W_{\lambda,\mu}}{\partial \lambda} = -(Z_1 - P_\psi) = 0,\ \frac{\partial W_{\lambda,\mu}}{\partial \mu} = \frac{1}{2}(\gamma^2 - 1) = 0$$

By the first relation in (2.1), $\omega = \mu\gamma$, and the steady motions are rotations at constant angular velocity λ. By the second and fourth relation of (2.1), we have

$$-\lambda(\lambda I\gamma + \mathbf{K}) + \mathbf{U}_\gamma + \mu\gamma = 0,\ \gamma^2 - 1 = 0 \tag{2.2}$$

Thus

$$\mu = \lambda[\lambda I(\gamma) + (\mathbf{K},\gamma)] - (\gamma, \mathbf{U}_\gamma)$$

where, by the third relation of (2.1)

$$\lambda(I(\gamma) + (\mathbf{K},\gamma) = P_\psi$$

Consequently

$$\lambda = (P_\psi - (K,\ \gamma))/I(\gamma)$$

where $I(\gamma) = I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2$ is the moment of inertia of the body relative to the γ axis. Then, by (2.2)

$$-\frac{[P_\psi - (\mathbf{K},\gamma)]^2}{I^2(\gamma)} I\gamma - \frac{P_\psi - (\mathbf{K},\gamma)}{I(\gamma)}\mathbf{K} + U_\gamma + \mu\gamma = 0 \tag{2.3}$$

In other words, the steady motions are determined as critical points of the effective potential

$$W_\mu = \frac{[P_\psi - (\mathbf{K},\gamma)]^2}{2I(\gamma)} + U(\gamma) + \frac{1}{2}\mu(\gamma^2 - 1) \tag{2.4}$$

In the general case, it is quite difficult to determine all the solutions of system (2.2), (2.3) for the potential (1.1). We will look for solutions in which the vector $\mathbf{K}$ is collinear with the vector $\gamma + \mathbf{K} = k\gamma$. Then, by (1.1), the equation for the steady motions may be written as

$$-(\gamma, \mathbf{U}_\gamma)\gamma + \mathbf{U}_\gamma = 0$$

This equation coincides with the equations for steady motions in the case of a vanishing gyrostatic moment. Consequently, if the condition $\mathbf{K} = k\gamma$ holds in steady motions of the gyrostat, the axes of rotation coincide with those of a single rigid body in the case of a vanishing gyrostatic moment; hence [1] there exist only rotations whose axes pass either through a vertex of the octahedron or through the centre of one of its faces.

Analogous results are valid for any other regular polyhedra with equal masses at the vertices.

3. STABILITY OF STEADY MOTIONS

To investigate the stability of the steady motions we have found, we will investigate the second variation of the function W_μ over the linear manifold

$$\delta Z_1 = (\gamma,\ \delta\gamma) = 0$$

as to whether it is of fixed sign.

We will consider motions in which one vertex of the body faces the attracting centre. One such motion corresponds to the orientation $\gamma = (0, 0, 1)$. Then

$$\delta Z_1 = 0 \Leftrightarrow \delta\gamma_3 = 0,\ 2\delta^2 W_\mu = \varkappa[(\delta\gamma_1)^2 + (\delta\gamma_2)^2]$$

$$\varkappa = -\frac{6fmMR^2a^2}{(R^2 + a^2)^{5/2}} - fmMRa\left[\frac{1}{|R + a|^3} - \frac{1}{|R - a|^3}\right] + \frac{P_\psi k - k^2}{I} > 0$$

We know [2] that when $\mathbf{K} = 0$ and $R > a$ the motion is stable in the secular sense. However, if $|\mathbf{K}|$ is sufficiently large, both Poincaré coefficients become negative and the instability degree is equal to two.

There exist motions such that

$$\gamma = (1,1,0)/\sqrt{2} = \gamma_{1-2}$$

In these motions the body turns one of its edges to the attracting centre. Then

$$\delta Z_1 = 0 \Leftrightarrow \{\delta\gamma : \delta\gamma_1 + \delta\gamma_2 = 0\}, \ 2\delta^2 W_\mu = \varkappa_1(\delta\gamma_1)^2 + \varkappa_3(\delta\gamma_3)^2$$

and the stability conditions are

$$\varkappa_1 = \frac{\partial^2 U(\gamma_{1-2})}{\partial\gamma_1^2} + \frac{P_\psi k - k^2}{I} - \sqrt{2}\frac{\partial U(\gamma_{1-2})}{\partial\gamma_1} > 0$$

$$\varkappa_3 = \frac{\partial^2 U(\gamma_{1-2})}{\partial\gamma_3^2} + \frac{P_\psi k - k^2}{I} - \sqrt{2}\frac{\partial U(\gamma_{1-2})}{\partial\gamma_3} > 0$$

There exist motions such that

$$\gamma_{1-3} = (1,1,1)/\sqrt{3}$$

In these motions the body turns one of its faces to the attracting centre. Then

$$\delta Z_1 = 0 \Leftrightarrow \{\delta\gamma : \delta\gamma_1 + \delta\gamma_2 + \delta\gamma_3 = 0\}, \ 2\delta^2 W_\mu = 3(A - B)[(\delta\gamma_1)^2 + \delta\gamma_1\delta\gamma_2 + (\delta\gamma_2)^2]$$

$$A = \frac{\partial^2 U(\gamma_{1-3})}{\partial\gamma_1^2} + \frac{P_\psi k - k^2}{I} + \frac{1}{\sqrt{3}}\frac{\partial U(\gamma_{1-3})}{\partial\gamma_1}, \quad B = \frac{\partial^2 U(\gamma_{1-3})}{\partial\gamma_1\partial\gamma_2} + \frac{(2P_\psi - k)^2}{3I}$$

If $A - B > 0$, the motion is stable in the secular sense; if $A - B < 0$, both Poincaré coefficients are negative and the instability degree is two.

It has been shown [2] that when $K = 0$ the number $A - B$ is negative if $R > 0$ and the instability degree of the corresponding motion is two. However, there is a region in the space (P_ψ, $\mathbf{K}$) in which this number is positive. Motions corresponding to this region turn out to be stable in the secular sense.

This research was supported by the International Association for the Promotion of Cooperation with Scientists from the Independent States of the Former Soviet Union (INTAS-93-1621-ext).

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Translated by D.L.